On K3 Correspondences

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Dedicated to Ammi, Abbu, Myda, Baji and Qudrat-Ullah Shahab with gratitude and love

Abstract

We consider relationships between families of K3 surfaces, in the context of string theory. An important ingredient of string theory also of interest in algebraic geometry is T-duality. Donagi and Pantev [DP] have extended the original duality on genus one fibred K3 surfaces with a section to the case of any genus one fibration, via a Fourier-Mukai transform. We investigate possibilities of extending this result to the more general case of non fibred K3s, in the context of Căldăraru's conjecture [C].

We also provide an explicit construction of a quasi-universal sheaf associated to the classical Mukai correspondence. (see [M1]).

1 Context and Background

Calabi–Yau manifolds A class of geometric spaces of interest to both mathematicians and physicists are Calabi–Yau (CY) manifolds.

Definition 1.1. *A CY manifold is a complex manifold X with a non vanishing top dimensional global holomorphic form and with $H^i(X, \mathcal{O}) = 0$ for $0 < i < n$, where $n = \dim_{\mathbb{C}} X$.*

Note: Some definitions in the literature omit the second condition.

Two dimensional CY manifolds are called K3 surfaces. There are dualities between different string theories that induce highly nontrivial relations amongst the geometric structures of the associated CY manifolds. One such duality is mirror symmetry, to which Candelas et al [CDFKM], Strominger, Yau and Zaslow [SYZ] and, most recently, Gross and Seibert [GS] have made major contributions.

Lately there has also been much interest in the derived category of twisted sheaves on CYs and dualities between them. We give the definition of a 'twisted sheaf.'

Definition 1.2. *Given $\theta \in \mathrm{H}^2(X, \mathcal{O}^*)$ and $\{U_i\}$ a sufficiently fine open cover of X, let θ_{ijk} be a Cech cocycle representative of θ. Then a θ-twisted sheaf on X is a collection $\{F_i\}, \{\phi_{ij}\}$ where F_i is a sheaf on U_i and ϕ_{ij} are isomorphisms relating F_i and F_j on $U_i \cap U_j$, such that $\phi_{ij}\phi_{jk}\phi_{ki} = \theta_{ijk}$ on $U_i \cap U_j \cap U_k$.*

Căldăraru has posed a beautiful conjecture relating equivalences between derived categories of twisted sheaves on K3 surfaces. Before we state the conjecture we give some facts about K3 surfaces. The second cohomology of a K3 surface is generated (over $\mathbb{Q}$) by the direct sum of its Picard lattice $\mathrm{Pic}(X)$ and its transcendental lattice T_X. The elements of the Picard lattice are the algebraic classes (Chern classes of holomorphic vector bundles on X). The lattice T_X is equal to the orthogonal complement of $\mathrm{Pic}(X)$ in $H^2(X, \mathbb{Z})$. The Brauer group $\mathrm{Br}(X)$ of an algebraic K3 surface X is the étale cohomology group $H^2_{\text{ét}}(X, \mathcal{O}^*)$. For an algebraic K3 surface X,

$$\mathrm{Br}(X) \cong T_X^\vee \otimes \mathbb{Q}/\mathbb{Z} \quad \text{(see [C])}.$$

Conjecture 1.1. *Let X and Y be algebraic K3 surfaces with transcendental lattices T_X and T_Y, and $\alpha \in \mathrm{Br}\, X$ and $\beta \in \mathrm{Br}\, Y$. Then the following are equivalent:*

1. *There exists an equivalence $\mathcal{D}^b(X, \alpha) \to \mathcal{D}^b(Y, \beta)$ of the derived categories of α-twisted sheaves on X to β-twisted sheaves on Y.*

2. *There exists a Hodge isometry $\ker \alpha \to \ker \beta$ between the sublattices of finite index $\ker \alpha \subset T_X$ and $\ker \beta \subset T_Y$.*

Orlov (see [O])has proved the case when both α and β are trivial. Căldăraru has proved the case for certain K3 surfaces where one of α or β is non trivial. His result is a genralisation of Mukai's theorem [M1].

Most recently Donagi and Pantev [DP] have proved the conjecture for genus one fibred K3 surfaces. Their result is a bit more general since only one of the K3s has to be algebraic.

Although these provide numerous examples of K3 surfaces for which the conjecture is true, to date, in its full generality, the conjecture stands unanswered.

One of the results in my thesis [K] which is a step towards an answer to this question is the following theorem about Hodge cycles on products of K3 surfaces.

Theorem 1.1. *There exists a 19-dimensional family $\mathcal{F}$ of pairs (M, Y), where M is a double cover of $\mathbb{P}^2$, Y is a degree 8 surface in $\mathbb{P}^5$ and $(T_M)_{\mathbb{Q}}$*

is Hodge isometric to $(T_Y)_{\mathbb{Q}}$*. The generic pairs* (M, Y) *in* $\mathcal{F}$ *are nonfibred K3 surfaces.*

The Hodge isometry between $T_{M\mathbb{Q}}$ and $T_{Y\mathbb{Q}}$ is induced by an algebraic cycle on the product $Z \in H^{2,2}(M \times Y, \mathbb{Q})$. This cycle is often referred to as the **isogeny cycle** or a **correspondence**.

The main point here is that for a generic pair (M, Y) in the above family, the isometry induced by the isogeny cycle does not restrict to an embedding of the integral lattice T_M into T_Y or the other way around. Note that in this respect the family above differs from Mukai's family of corresponding pairs (M, Y), where T_Y embeds in T_M as a sublattice of index 2.

Based on the idea behind this example we construct infinitely many families of isogenous K3s. These families are natural testing grounds for the conjecture.

In Section 2 we will list some preliminary facts about K3 surfaces In Section 3 we will explain the Donagi-Pantev result with examples. In Section 4 we will illustrate the main idea with a couple of examples, before giving the general construction. In Section 5 we will give the explicit construction of the quasi universal sheaf associated to the classical Mukai correspondence.

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2 K3 Surfaces

We state some facts about K3 surfaces which we will use later.

2.1 Topological and analytical invariants

Notation. *Let* X *be a compact connected complex surface with the cup product form* $\langle , \rangle$ *on* $\mathrm{H}^2(X, \mathbb{C})$*.*

Let $L = -E_8 \oplus -E_8 \oplus H \oplus H \oplus H$ *where* E_8 *is the root lattice of the exceptional Lie group* E_8*.* H *is the hyperbolic lattice with intersection form* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$L_{\mathbb{C}} := L \otimes \mathbb{C}$ with $\langle , \rangle$ extended $\mathbb{C}$ bilinearly. For $\omega \in L_{\mathbb{C}}$, we denote by $[\omega]$ in $\mathbb{P}(L_{\mathbb{C}})$ the corresponding line and set $\Omega = \{[\omega] \in \mathbb{P}L_{\mathbb{C}} : \langle \omega, \omega \rangle = 0, \langle \omega, \overline{\omega} \rangle > 0\}$.

Definition 2.1. *A K3 surface is a (possibly singular) surface with trivial canonical bundle and $b_1(X) = 0$.*

Proposition 2.1. *Let X be a smooth K3 surface, then $c_1(X) = 0$, $c_2(X) = 24$ and $\mathrm{H}^2(X, \mathbb{Z})$ equipped with the cup product pairing is isometric to the lattice L.*

The Hodge numbers of a K3 surface are completely determined.

Proposition 2.2. *Let X be a smooth K3 surface. Then $h^{0,1}(X) = h^{1,0}(X) = h^{2,1}(X) = h^{1,2}(X) = 0$, $h^{0,2}(X) = h^{2,0}(X) = 1$, $h^{1,1}(X) = 20$.*

The proofs involve Kodaira-Serre duality and Riemann-Roch for surfaces. (For more details, see [BPV], Chapter VIII.)

If X and Y are surfaces, an isomorphism of $\mathbb{Z}$-modules

$$\mathrm{H}^2(X, \mathbb{Z}) \rightarrow \mathrm{H}^2(Y, \mathbb{Z})$$

is called a **Hodge isometry** if it preserves the cup product $\langle , \rangle$, and its $\mathbb{C}$-linear extension preserves the Hodge decomposition.

We will be interested in the Hodge isometries over $\mathbb{Q}$.

The Neron-Severi group of X, denoted by N_X, is equal to $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$. In the case X is a smooth K3 surface, N_X is actually isomorphic to $\mathrm{Pic}(X)$. The transcendental lattice of X is $T_X := N_X^{\perp}$ in $\mathrm{H}^2(X, \mathbb{Z})$.

Both N_X and T_X are primitive sublattices of $\mathrm{H}^2(X, \mathbb{Z})$, but are generically non-unimodular.

Given two K3 surfaces X and Y, a cycle $Z \in \mathrm{H}^2(X)_{\mathbb{Q}} \otimes \mathrm{H}^2(Y)_{\mathbb{Q}}$ and projections p_X, p_Y onto the first and second factor respectively, there exists an associated mapping f_Z,

$$\begin{array}{cccc} f_Z & : & \mathrm{H}^2(X, \mathbb{Q}) & \longrightarrow & \mathrm{H}^2(Y, \mathbb{Q}) \\ & & t & \mapsto & p_{Y*}(p_X^*(t) \cdot Z) \end{array}$$

Sometimes we will denote f_Z by Z, for ease of notation.

We use the following definition. See [M2], Definition 1.7.

Definition 2.2. *An algebraic cycle Z in $(\mathrm{H}^2(X) \otimes \mathrm{H}^2(Y))_{\mathbb{Q}}$ is an **isogeny** (or a **correspondence**), if the homomorphism $Z : \mathrm{H}^2(X, \mathbb{Q}) \rightarrow \mathrm{H}^2(Y, \mathbb{Q})$ is an isometry.*

Since $\mathrm{H}^2(X,\mathbb{Q}) = (N_X \oplus T_X)_{\mathbb{Q}}$ and $\mathrm{H}^2(Y,\mathbb{Q}) = (N_Y \oplus T_Y)_{\mathbb{Q}}$,

$$\mathrm{H}^2(X,\mathbb{Q}) \otimes \mathrm{H}^2(Y,\mathbb{Q}) = [(N_X \otimes N_Y) \oplus (N_X \otimes T_Y) \oplus (T_X \otimes N_Y) \oplus (T_X \otimes T_Y)]_{\mathbb{Q}}.$$

The vector space $((N_X \otimes T_Y) \oplus (T_X \otimes N_Y)$ does not contain cohomology classes of type $(2,2)$.

Therefore $Z = Z_N + Z_T$, where $Z_N \in N_X \otimes N_Y$ and $Z_T \in T_X \otimes T_Y$. The Z_N component is always algebraic, hence a Hodge cycle Z is algebraic if and only if Z_T is algebraic.

2.2 Period mapping

Given a smooth K3 surface X, $\mathrm{H}^2(X,\mathbb{Z})$ is isometric to the lattice L. The choice of an isometry $\varphi : \mathrm{H}^2(X,\mathbb{Z}) \to L$ determines a line in $L_{\mathbb{C}}$, spanned by the $\varphi_{\mathbb{C}}$-image of the nowhere vanishing holomorphic 2-form ω_X.

The identity $\langle \omega, \omega \rangle = 0$ and the inequality $\langle \omega, \overline{\omega} \rangle > 0$ imply that this line considered as a point in $\mathbb{P}(L_{\mathbb{C}})$ belongs to Ω. Recall that Ω is the period domain, defined earlier in Section 2.1. This point is called the **period point** of a marked K3 surface (X, φ).

There are two main theorems regarding marked K3 surfaces.

Theorem 2.1 (Torelli Theorem). *Two K3 surfaces X,Y are isomorphic if and only if there exists a Hodge isometry $\varphi : \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(Y,\mathbb{Z})$.*

Theorem 2.2. *All points of Ω occur as period points of marked K3 surfaces.*

(For proofs, see [BPV], Chapter VIII.)

Definition 2.3. *A family $p : \mathcal{X} \to \Delta$ is said to be a **marked deformation** if there exists a trivialization $\varphi : R^2 p_* \mathbb{Z}_{\mathcal{X}} \to L$.*

Associated to a marked deformation of K3 surfaces we have the period mapping.

Definition 2.4. *Suppose we have a marked deformation of K3 surfaces, the associated **period mapping** $\tau : \Delta \to \Omega$ is defined by sending $t \in \Delta$ to the complex line $\varphi_{\mathbb{C},t}(\mathrm{H}^{2,0}(\mathcal{X}_t))$ in $L_{\mathbb{C}}$, and then taking the corresponding point in the projective space $[\varphi_{\mathbb{C},t}(\mathrm{H}^{2,0}(\mathcal{X}_t))] \in \mathbb{P}(L_{\mathbb{C}})$.*

Notation. *Given a family $p : \mathcal{X} \to B$ of K3 surfaces, let $\Omega_{\mathcal{X}}$ denote the image of $\mathcal{X}$ under the associated period mapping τ.*

The map τ is holomorphic. We will refer to the image of τ in Ω as the period points. (Occasionally, we will consider Ω in $L_{\mathbb{C}}$ instead of $\mathbb{P}(L_{\mathbb{C}})$, and a generator of $\varphi_{\mathbb{C}}(\mathrm{H}^{2,0}(\mathcal{X}_t))$ as the period vector.)

Definition 2.5. *A deformation $p : \mathcal{X} \to B$, of a K3 surface $X_0 := p^{-1}(b_0)$ is called (locally) universal, if (locally) every other deformation $p' : \mathcal{X}' \to B'$ is obtained as the pullback from $\mathcal{X}$ by a suitable analytic map $f : B' \to B$, with $f(b'_0) = f(b_0)$.*

We have the following theorem due to Kodaira.

Theorem 2.3. *Given a K3 surface X_0, there exists a locally universal deformation $p : \mathcal{X} \to B$ of the surface X_0. The associated period mapping $\tau : B \to \Omega$ is a local isomorphism.*

The family of deformations of a K3 surface is unobstructed and in particular 20 dimensional. The K3 surface corresponding to a generic point of Ω is not algebraic. The family of algebraic K3 surfaces corresponds to a dense countable union of hypersurfaces of the period domain. See [BPV] for more details.

3 Genus one fibrations

In this section we explain the Donagi-Pantev result with its applications to some examples.

3.1 Examples

We first give some definitions.

Definition 3.1. *A* ***genus one fibered*** *K3 is a K3 surface X, which admits a morphism to $\mathbb{P}^1$, such that the generic fiber of this morphism is a genus one curve.*

Given a curve C in X, the fibre degree of C is defined to be the intersection number $C \cdot f$ where f is the generic fibre. Consider the fibre degree of all curves in X. Then we have the following definition.

Definition 3.2. *Given a genus one fibred algebraic K3 surface X the smallest positive fibre degree is called the* ***index*** *of X.*

Definition 3.3. *A* ***multisection*** *of a fibration is a smooth curve in X which intersects each fiber in finitely many points.*

If X admits a section then the index is equal to 1 and X is called an **elliptic fibration**. In the case of algebraic K3s the index can be viewed as a measure of how far away the surface is from being an elliptic fibration.

The following are examples of algebraic families of genus one fibrations.

Definition 3.4. *Let $\mathcal{M}$ denote the 19 dimensional family of K3 surfaces which can be realized as double covers of $\mathbb{P}^2$ branched along a sextic curve B.*

Note. *Here $\mathcal{M}$ actually stands for the parameter space of all such K3s. We will often refer to the K3 surface M corresponding to a generic point in $\mathcal{M}$ as the generic element of $\mathcal{M}$.*

1. *Let $\mathcal{M}_\alpha^s$ denote the codimension one subfamily of $\mathcal{M}$ consisting of double covers of $\mathbb{P}^2$, branched along a sextic curve with at least one ordinary double point. This contains a Zariski open set U_α of sextics with exactly one node.*

2. *Let $\mathcal{M}_\alpha$ be the resolution of this family. That is, there is a map $\psi : \mathcal{M}_\alpha \to \mathcal{M}_\alpha^s$ where $\psi : M_\alpha \to M_\alpha^s$ is the resolution of the generic element M_α^s.*

Using the adjunction formula, Serre duality and the Riemann-Roch theorem for surfaces, we get the following results.

Lemma 3.1. *The generic element M_α of $\mathcal{M}_\alpha$ is a genus one fibered K3 surface with index 2. The Picard group of M_α is generated by the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ and the class of the exceptional curve.*

Lemma 3.2. *Consider the zero locus of a homogeneous equation of bi-degree $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. It is a K3 surface, which is a double cover of $\mathbb{P}^2$ branched along a sextic, as well as a genus one fibration of index 3.*

Let M_β denote the surface corresponding to the class $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$.

Projection onto the first factor gives a morphism $\pi_1 : M_\beta \to \mathbb{P}^1$, with the generic fiber a smooth genus one curve corresponding to a cubic curve in $\mathbb{P}^2$.

Projection onto the second factor gives a morphism $\pi_2 : M_\beta \to \mathbb{P}^2$, which is generically a $2:1$ map and realises M_β as a double cover of $\mathbb{P}^2$.

Definition 3.5. *Let M_β be a surface in $\mathbb{P}^1 \times \mathbb{P}^2$ with divisor class $(2,3)$. Let $F := (1,0)_{|M_\beta}$, $D := (0,1)_{|M_\beta}$, where $(1,0)$ and $(0,1)$ denote the pullbacks of $\mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_{\mathbb{P}^2}(1)$ respectively.*

Then $\mathrm{Pic}(M_\beta) = \mathbb{Z}\, D \oplus \mathbb{Z}\, F$ where $\langle D, D\rangle = 2, \langle F, F\rangle = 0, \langle D, F\rangle = 3$.

The generic curve in $|D|$ is a multisection of fiber degree 3. Therefore M_β is a genus one fibration of index 3.

Let $\mathcal{M}_\beta$ denote the family of all K3 surfaces obtained in this way. The generic element M_β embeds in $\mathbb{P}^5$ as a surface of degree 8.

Similarly we Let $\mathcal{Y}$ denote the 19 dimensional family of degree 8 K3 surfaces in $\mathbb{P}^5$. The generic element Y of $\mathcal{Y}$ is a complete intersection of three quadric hypersurfaces.

The family $\mathcal{M}_\beta$ is a codimension one subfamily of $\mathcal{Y}$, and $\mathcal{M}_\alpha$ is a codimension one subfamily of $\mathcal{M}$.

3.2 Ogg-Shafarevich Theory

Given a smooth genus one fibration $X \rightarrow B$, there is a naturally associated elliptic fibration $J \rightarrow B$, called the relative Jacobian of X. More precisely we have the following definition.

Definition 3.6. *Let $p_X : X \rightarrow B$ be a genus one fibration. Fix a relatively ample sheaf $\mathcal{O}_X(1)$. Let $p_J : J \rightarrow B$ be the relative moduli space of semistable torsion-free sheaves of rank 1, degree 0 on the fibers of X. The fibration $p_J : J \rightarrow B$ is called the relative Jacobian of $X \rightarrow B$.*

Note. *We will sometimes denote $p_J : J \rightarrow B$, by $J(X)$.*

For existence, see [FM].

An elliptic fibration $p_J : J \rightarrow B$ can be the relative Jacobian of different genus one fibrations. One way to distinguish them is by the index. In the case of algebraic K3 surfaces, the index is always finite. For example the generic element $M_\alpha \in \mathcal{M}_\alpha$ is a genus one fibred K3 of index 2, while $M_\beta \in \mathcal{M}_\beta$ has index 3.

Definition 3.7. *There are two natural invariants associated to a genus one fibration $p_X : X \rightarrow B$. Let f_b denote the fiber over $b \in B$.*

1. ***j-function:*** *The j-function of X denoted by j_X is a function $j_X : B \rightarrow \mathbb{P}^1$ defined by*
$$j_X : b \rightarrow j(f_b)$$
, where $j(f_b)$ is the j-invariant of f_b.

2. ***Monodromy Presentation**: Let $\{b_1, \ldots, b_k\}$ (also known as the discriminant) denote the points over which the fibers are singular. The monodromy presentation of X is a homomorphism*
$$\rho_X : \pi_1(B - \{b_1, \ldots, b_k\}) \rightarrow SL(2, \mathbb{Z}).$$

Let $J \rightarrow \mathbb{P}^1$ be an elliptic surface (i.e. it has a section). Then associated to J there is another elliptic surface $\overline{J} \rightarrow \mathbb{P}^1$, called the **Weierstrass model** of J. It is uniquely determined up to fibre preserving automorphisms of J.

We have the following theorem by Friedman on the moduli space of elliptic K3 surfaces (See [FM], Proposition 4.3).

Theorem 3.1. *Let $J \to \mathbb{P}^1$ be an elliptic K3 surface. Then $\overline{J}$ is determined by the choice of two sections $g_2 \in \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(8))$ and $g_3 \in \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(12))$, modulo the action of $\mathbb{C}^*$.*

We identify two elliptic surfaces over $\mathbb{P}^1$ if there is an isomorphism between them covering some automorphism of the base, $\mathbb{P}^1$. The moduli space of such surfaces(denoted by $\mathcal{J}$) has dimension 18. Let $U \subset \mathcal{J}$ be the set corresponding to the pairs (g_2, g_3) such that

1. *$g_2^3 - 27g_3^2$ is a nonzero section of $\mathcal{O}(24)$,*
2. *For all $p \in \mathbb{P}^1$, $\min\{3v_p(g_2), 2v_p(g_3)\} < 12$, where v_p denotes the order of vanishing of the corresponding section at p.*

Then U is Zariski open, and non-empty. Conditions 1) and 2) ensure that the generic fiber of $p_J : J \to \mathbb{P}^1$ is smooth and that J has at most ordinary double points.

The generic point $[J]$ in the moduli space corresponds to a smooth surface J with 24 nodal fibers. Then j_J has degree 24 (j_J maps the discriminant to the point ∞) and the monodromy presentation is a homomorphism

$$\rho_J : \pi_1(\mathbb{P}^1 - \{b_1, \ldots, b_{24}\}) \to SL(2, \mathbb{Z}).$$

If X is a genus one fibered K3 surface, then X has the same j-function as its relative Jacobian $J(X)$, and its monodromy presentation ρ_X is isomorphic to ρ_J, via conjugation.

The classical theory of elliptic fibrations has been developed by Ogg and Shafarevich. For details of the technical aspects, see [DG].

The main idea is that given an elliptic fibration $p_J : J \to B$, the set of all genus one fibrations $p_X : X \to B$ whose relative Jacobian is isomorphic to J is in one-to-one correspondence with the Tate-Shafarevich group $\mathrm{Sh}(J)$ of J.

Definition 3.8. *Given an elliptic fibration $p_J : J \to B$, the Tate-Shafarevich group $\mathrm{Sh}(J)$ is defined to be $\mathrm{H}^1(B, J^\#)$ where $J^\#$ is the sheaf of local sections of $p_J : J \to B$.*

As mentioned earlier, all elements of $\mathrm{Sh}(J)$ have the same j-function and isomorphic monodromy presentation.

This makes sense in any topology. Of interest to us are the étale and analytic topologies. However, the algebraic genus one fibrations associated to J correspond only to $\mathrm{H}^1_{\text{ét}}(B, J^{\#})$. (See [DP]).

The Ogg-Shafaravich Theory of elliptic fibrations is defined for any dimension. In the case that J is an elliptic K3 surface, we get a simple characterization of $\mathrm{Sh}(J)$.

Lemma 3.3. *Let $J \rightarrow \mathbb{P}^1$ be an elliptic K3 surface. Then*

$$\mathrm{Sh}(J) = \mathrm{Br}(J) = T_J^{\vee} \otimes \mathbb{Q}/\mathbb{Z}.$$

Proof. See [DG]. □

Căldăraru shows in his thesis that J, viewed as a relative moduli space of sheaves on X, is not a fine moduli space. In fact, if the surface X corresponds to the element $\alpha \in \mathrm{Br}(J)$, then there exists a $p_2^*(\alpha)$-twisted Poincare sheaf $\mathcal{E}$ on $X \times J$, where p_1, p_2 are the first and the second projections.

This twisted Poincare sheaf $\mathcal{E}$ induces a map on the level of cohomology from $\mathrm{H}^*(X, \mathbb{Q}) \rightarrow \mathrm{H}^*(J, \mathbb{Q})$ called the Fourier-Mukai Transform which is a Hodge isometry of the extended Mukai product (see Section 5).

As a result, we get a short exact sequence

$$0 \rightarrow T_X \rightarrow T_J \xrightarrow{\alpha} \mathbb{Z}/n\mathbb{Z} \rightarrow 0.$$

In other words, T_X embeds in T_J as the kernel of α.

Let X_α, Y_β be two genus one fibrations associated to the elliptic fibration J. Then the index of T_{X_α} (resp. T_{Y_β}) is equal to $[T_J : T_{X_\alpha}]$ (resp. $[T_J : T_{Y_\beta}]$).

Let $\overline{\beta} := \beta_{|T_{X_\alpha}}$ and $\overline{\alpha} := \alpha_{|T_{Y_\beta}}$. Then $\ker \overline{\beta} = \ker \alpha \cap \ker \beta = \ker \overline{\alpha}$. So $\ker \overline{\beta}$ is Hodge isometric to $\ker \overline{\alpha}$. Donagi and Pantev have proved the following interesting case of Căldăraru's conjecture.

Theorem 3.2. *Let J/S be an elliptic K3 surface with a section, and let $\alpha, \beta \in Br(J)$. Identifying $Br(J)$ with the Tate-Shafarevich group of J yields genus one fibered K3 surfaces X_α, Y_β (in general without a section) which correspond to α, β respectively. Viewing J as a moduli space of stable sheaves on X_α and Y_β gives surjections $Br(J) \rightarrow Br(X_\alpha)$ and $Br(J) \rightarrow Br(Y_\beta)$; let $\bar{\beta}$ be the image of β in $Br(X_\alpha)$ and $\bar{\alpha}$ be the image of α in $Br(Y_\beta)$. Then there exists an equivalence of derived categories of twisted sheaves $\mathcal{D}^b(X_\alpha, \bar{\beta})$ and $\mathcal{D}^b(Y_\beta, \bar{\alpha})$.*

We will now apply the Donagi-Pantev result to the families $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ discussed earlier. They are 18 dimensional algebraic families of genus one fibered K3 surfaces, whose generic elements M_α, M_β are not elliptically fibred. Instead they have multisections of index 2 and 3 respectively.

Definition 3.9. *Let U_α (resp. U_β) in $\mathcal{M}_\alpha$ (resp. $\mathcal{M}_\beta$) denote the Zariski open subset corresponding to smooth genus one fibrations with 24 nodal fibers.*

Then we have the following theorem:

Theorem 3.3. *There exists a finite to finite correspondence between the families $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$, over the open sets $U_\alpha \subset \mathcal{M}_\alpha, U_\beta \subset \mathcal{M}_\beta$. That is for each element $M_\alpha \in \mathcal{M}_\alpha$, there exist finitely many elements $\{M_\beta^j\}_{j=1}^l \subset \mathcal{M}_\beta$, with isomorphic relative Jacobian.*

Proof. The main idea is to show the classifying morphisms from U_α and U_β to $\mathcal{J}$ are dominant maps between algebraic varieties of the same dimension. Then it follows that they have finite fibres.

As in Theorem 3.1, let $U \subset \mathcal{J}$ denote the Zariski open irreducible subset corresponding to smooth elliptic fibrations with 24 nodal fibers. Let $p_J : J \to \mathbb{P}^1$, be a representative of $[J] \in U$. Let j_J denote the j-function of J, and let ρ denote the monodromy representation of J.

Let $\varphi_\alpha : \mathcal{M}_\alpha \dashrightarrow \mathcal{J}$ denote the rational map sending a fiber M_α to the isomorphism class of its relative Jacobian, $[J(M_\alpha)]$. Similarly, let $\varphi_\beta : \mathcal{M}_\beta \dashrightarrow \mathcal{J}$ denote the classifying morphism for the family $\mathcal{M}_\beta$.

Since U_α is an irreducible Zariski open subset of $\mathcal{M}_\alpha$, the image of U_α under φ_α is an irreducible subvariety of $U \subset \mathcal{J}$.

Fix $[J] \in \varphi_\alpha(U_\alpha)$.

Then all the elements in $\varphi_\alpha^{-1}[J]$ have the same j-function and isomorphic monodromy representation. The set $\mathrm{Hom}(\pi(\mathbb{P}^1 - \{d_1, \ldots, d_{24}\}), SL(2, \mathbb{Z}))$ is discrete hence so is the orbit of ρ_J under conjugation, which is the set of all representations isomorphic to ρ_J.

It follows that the preimage $\varphi_\alpha^{-1}[J]$ in U_α is discrete. So the image of U_α in $\mathcal{J}$ is 18 dimensional, and hence an irreducible Zariski open subset (i.e. φ_α is dominant). The same argument applies to φ_β.

Since $\varphi_{\alpha|U_\alpha}, \varphi_{\beta|U_\beta}$ are algebraic maps between irreducible varieties of the same dimension, they have finite fibers. So given any point $[J]$ in the image of φ_α and φ_β there exist finitely many elements $\{M_\alpha^i\}_{i=1}^k, \{M_\beta^j\}_{j=1}^l \in \mathcal{M}_\alpha, \mathcal{M}_\beta$ respectively with their relative Jacobian isomorphic to J. $\square$

This then gives us 18 dimensional loci in $\mathcal{M}_\alpha \times \mathcal{M}_\beta$ of pairs (M_α, M_β) with isomorphic jacobians. Recall that T_{M_α} and T_{M_β} embed in T_J as $\ker \alpha$, $\ker \beta$ respectively for some $\alpha, \beta \in \mathrm{Br}(J)$ and that $\ker \beta_{|T_{M_\alpha}} \cong \ker \alpha_{|T_{M_\beta}}$. The Donagi-Pantev result then implies that

$$\mathcal{D}(M_\alpha, \overline{\beta}) \cong \mathcal{D}(M_\beta, \overline{\alpha}).$$

So we have an explicit example where Căldăraru's Conjecture holds.

There is a natural corollary to this theorem.

Corollary 3.1. *The correspondence of Theorem 3.3 above implies that for each i and j, there exists an algebraic cycle $Z^{ij} \in \mathrm{H}^{2,2}(M^i_\alpha \times M^j_\beta, \mathbb{Q})$ such that*

$$Z^{ij} : \mathrm{H}^2(M^i_\alpha, \mathbb{Q}) \to \mathrm{H}^2(M^j_\beta, \mathbb{Q})$$

is a Hodge isometry.

Proof. Since J is the relative moduli space of sheaves on M^i_α and M^j_β for each i, j ($1 \leq i \leq k$, $1 \leq j \leq l$), there exist maps

$$0 \to T_{M^i_\alpha} \to T_J \to \mathbb{Z}/2\mathbb{Z} \to 0$$

,

$$0 \to T_{M^j_\beta} \to T_J \to \mathbb{Z}/3\mathbb{Z} \to 0.$$

This implies that there exists a Hodge isometry $\psi^{ij} : (T_{M^i_\alpha})_{\mathbb{Q}} \to (T_{M^j_\beta})_{\mathbb{Q}}$. Let Z^{ij} denote the corresponding element of $(T_{M^i_\alpha} \otimes T_{M^j_\beta})_{\mathbb{Q}}$.

By the following recent theorem of Mukai (see [M3]), it follows that Z^{ij} is algebraic.

Theorem 3.4. *Let X and Y be algebraic K3 surfaces. Suppose we are given a map $f : \mathrm{H}^2(X, \mathbb{Q}) \to \mathrm{H}^2(Y, \mathbb{Q})$ which is a Hodge isometry. Then the associated cycle Z_f is algebraic.*

□

Our goal is to find other loci consisting of K3 surfaces exhibiting a similar correspondence. These will provide a suitable framework for extending the Donagi-Pantev result to non fibred K3 surfaces.

In the next section we will exhibit such correspondences between non-fibred K3 surfaces.

We remark here that the Donagi-Pantev theorem can be viewed as a statement about T-duality on gerbes over K3s.

4 Families of correspondences

In this section we show the existence of infinitely many families of pairs of corresponding K3 surfaces. We first start with some examples.

4.1 Examples

Theorem 3.3 implies that there exists an 18 dimensional locus $\mathcal{F} \subset \mathcal{M}_\alpha \times \mathcal{M}_\beta$ with the following properties:

1. Given (M_α, M_β) corresponding to points in $\mathcal{F}$, there exists an algebraic class Z in $\mathrm{H}^{2,2}(M_\alpha \times M_\beta, \mathbb{Q})$ such that the associated map $Z : \mathrm{H}^2(M_\alpha, \mathbb{Q}) \to \mathrm{H}^2(M_\beta, \mathbb{Q})$ is a Hodge isometry,

2. The pairs M_α, M_β correspond to elements $\alpha, \beta \in \mathrm{Br}(J)$ resp. where J is the associated relative Jacobian fibration. Recall that T_{M_α} embeds in T_J as $\ker \alpha$ and T_{M_β} embeds in T_J as $\ker \beta$.

Let $\overline{\alpha} := \beta|_{T_{M_\alpha}}, \overline{\beta} := \alpha|_{T_{M_\beta}}$, be the corresponding elements in $\mathrm{Br}(M_\alpha)$ and $\mathrm{Br}(M_\beta)$.

Then $\ker \overline{\alpha}$ is Hodge isometric to $\ker \overline{\beta}$, via Z.

We would like to extend the above isometry to other elements of $\mathcal{M} \times \mathcal{Y}$.

Theorem 4.1. *There exists a 19 dimensional family in $\mathcal{M} \times \mathcal{Y}$ of isogenous pairs (M, Y) such that $(T_M)_\mathbb{Q}$ is Hodge isometric to $(T_Y)_\mathbb{Q}$.*

Proof of Theorem 4.1. Consider a marked deformation of M_α inside the family $\mathcal{M}$. If we choose $U \subset \mathcal{M}$ small enough so that it is contractible then such a marking exists. For M a general element of U we get an inclusion
$i : T_{M_\alpha} \to T_M$.

Recall that $N_{M_\alpha} = \mathbb{Z}\, D \oplus \mathbb{Z}\, e$ where

$$\langle D, D\rangle = 2, \langle D, e\rangle = 0, \langle e, e\rangle = -2.$$

Since $N_M := \mathbb{Z}\, D$, $N_M^\perp$ in N_{M_α} is just $\mathbb{Z}\, e$. Hence we get an inclusion $i : T_{M_\alpha} \oplus \mathbb{Z}\, e \to T_M$ as a sublattice of index 2.

Next, consider a marked deformation of $\mathcal{M}_\beta$ in the family $\mathcal{Y}$. Let Y be the generic element. Let H denote the hyperplane class of Y. When restricted to a generic element M_β in the subfamily $\mathcal{M}_\beta$, $H = D + F$. As before $N_{M_\beta} = \mathbb{Z}\, D \oplus \mathbb{Z}\, F$, where

$$\langle D, D\rangle = 2, \langle D, F\rangle = 3, \langle F, F\rangle = 0.$$

Since $N_Y = \mathbb{Z}\, H$ and $H = D + F$ on a generic element M_β we have $N_Y^\perp = \mathbb{Z}\,(3H - 8F)$ in N_{M_β}. Let $r := 3H - 8F$, then we have an inclusion: $i : T_{M_\beta} \oplus \mathbb{Z}\, r \to T_Y$ as a sublattice of index 9.

According to Corollary 3.1 we already have a Hodge isometry between $(T_{M_\alpha})_{\mathbb{Q}}$ and $(T_{M_\beta})_{\mathbb{Q}}$ induced by a correspondence Z'_T. We would like to know when we can extend this isometry to other elements of the families $\mathcal{M}, \mathcal{Y}$.

Let $V := (T_M)_{\mathbb{Q}}, W := (T_Y)_{\mathbb{Q}}$.
Let $V' := (T_{M_\alpha})_{\mathbb{Q}}, W' := (T_{M_\beta})_{\mathbb{Q}}$.
It follows that $V = V' \oplus \mathbb{Q}\, e$, $W = W' \oplus \mathbb{Q}\, r$ and $Z'_T : V' \to W'$ is an isometry.

Consider the following element of $V^* \otimes W$,
$Z_T := Z'_T - \frac{1}{12} e \otimes r$. Then it is easy to check that Z_T induces an isometry between V and W.

Since T_M and T_Y are transcendental lattices of K3s, V and W both embed in $L_{\mathbb{Q}}$.

Now let $\omega_M \in V_{\mathbb{C}}$ be the associated period point of M. Since Z_T is a $\mathbb{Q}$ isometry, the image $Z_T(\omega_M)$ under its $\mathbb{C}$-linear extension lies in the period domain Ω.

Note. *We have the following relations*

$$\langle Z_T(\omega_M), Z_T(\omega_M)\rangle = 0$$

and

$$\langle \omega_M, \overline{\omega}_M\rangle = \langle Z_T(\omega_M), Z_T(\overline{\omega}_M)\rangle > 0$$

which are the relations satisfied by points in Ω.

By the surjectivity of the period mapping we get that $Z_T(\omega_M)$ is the period point ω_Y of some K3 surface Y. For this choice of Y, $Z_T : (T_X)_{\mathbb{Q}} \to (T_Y)_{\mathbb{Q}}$ becomes a Hodge isometry.

It is easy to extend this isometry to all of $\mathrm{H}^2(M)_{\mathbb{Q}}$. Consider the cycle $Z := \frac{1}{4} D \otimes H + Z_T \in \mathrm{H}^2(M, \mathbb{Q}) \times \mathrm{H}^2(Y, \mathbb{Q})$. Then Z maps the polarisation on M to a rational multiple of the polarisation on Y and restricts to the Hodge isometry $Z_T : (T_M)_{\mathbb{Q}} \longrightarrow (T_Y)_{\mathbb{Q}}$. Hence it induces a Hodge isometry

$$Z : \mathrm{H}^2(M, \mathbb{Q}) \longrightarrow \mathrm{H}^2(Y, \mathbb{Q}).$$

By Mukai's result it now follows that Z is algebraic. This proves Theorem 4.1.

□

Note. *The Hodge isometry* $Z_T : (T_M)_{\mathbb{Q}} \longrightarrow (T_Y)_{\mathbb{Q}}$ *is strictly defined over* $\mathbb{Q}$ *because its component* Z'_T *is strictly defined over* $\mathbb{Q}$.

4.2 General construction

We will now generalise the above example.

Lemma 4.1. *Let $\mathfrak{X}_{3k}, \mathfrak{X}_{3k+1}, \mathfrak{X}_{3k+2}$ denote the 19 dimensional families of algebraic K3s in $\mathbb{P}^{3k}, \mathbb{P}^{3k+1}, \mathbb{P}^{3k+2}$ respectively.*
Let $X_{3k} \in \mathfrak{X}_{3k}, X_{3k+1} \in \mathfrak{X}_{3k+1}, X_{3k+2} \in \mathfrak{X}_{3k+2}$ denote the general elements with polarisations $D_{3k}, D_{3k+1}, D_{3k+2}$ respectively. Then

$$\langle D_{3k}, D_{3k} \rangle = 6k - 2, \langle D_{3k+1}, D_{3k+1} \rangle = 6k, \langle D_{3k+2}, D_{3k+2} \rangle = 6k + 2.$$

We consider the following 18 dimensional families of genus one fibred K3s:

1. *Let $\mathfrak{J}_0$ be the family of quartic surfaces in $\mathbb{P}^3$ containing a line.*
2. *Let $\mathfrak{J}_1$ be the family of smooth K3s given by the intersection of a smooth cubic hypersurface with a nodal quadric.*
3. *Let $\mathfrak{J}_2$ be the family of genus one fibred K3s corresponding to the class of $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$.*

Then we have the following inclusions

$$\mathfrak{J}_0 \subset \mathfrak{X}_{3k}, \mathfrak{J}_1 \subset \mathfrak{X}_{3k+1}, \mathfrak{J}_2 \subset \mathfrak{X}_{3k+2}.$$

The proof follows from an application of the Riemann-Roch theorem to show that linear systems of the form $H+(k-1)E, k \geq 1$, where H represents the hyperplane class and E represents an elliptic curve, are very ample.

Theorem 4.2. *There are infinitely many pairs of integers (k, l) and (m, l), satisfying certain constraints for which there exist 19 dimensional loci of correspondences in $\mathfrak{X}_{3k} \times \mathfrak{X}_{3l+1}$ and $\mathfrak{X}_{3m+2} \times \mathfrak{X}_{3l+1}$. That is, there are 19 dimensional families of pairs (X_{3k}, X_{3l+1}) and (X_{3m+2}, X_{3l+1}) on which there exist algebraic correspondences inducing Hodge isometries: $Z_{3k,3l+1}$: $\mathrm{H}^2(X_{3k}, \mathbb{Q}) \to \mathrm{H}^2(X_{3l+1}, \mathbb{Q})$ and $Z_{3l+1,3m+2} : \mathrm{H}^2(X_{3m+2}, \mathbb{Q}) \to \mathrm{H}^2(X_{3l+1}, \mathbb{Q})$.*

Proof. The idea is the same as in Theorem 4.1. There exists a finite to finite correspondence between $\mathfrak{J}_0$ and $\mathfrak{J}_1$ (Theorem 3.3). Let $J_0 \in \mathfrak{J}_0$ and $J_1 \in \mathfrak{J}_1$ be genus one fibred K3s with isomorphic Jacobians. By Corollary 3.1 there exists an algebraic cycle $Z'_T \in (T_{J_0} \otimes T_{J_1})_{\mathbb{Q}}$ such that $Z'_T : (T_{J_0})_{\mathbb{Q}} \to (T_{J_1})_{\mathbb{Q}}$ is a Hodge isometry.

Next consider a marked deformation of $J_0 \in \mathfrak{X}_{3k}$. Then $T_{J_0} \to T_{X_{3k}}$ where X_{3k} is a generic fibre. Let D_{3k} denote the hyperplane class of X_{3k}. Similarly consider a marked deformation of $J_1 \in \mathfrak{X}_{3l+1}$. Again for X_{3l+1} a

generic element, $T_{J_1} \to T_{X_{3l+1}}$ and the polarisation D_{3l+1} is the orthogonal complement of $T_{X_{3l+1}}$

Consider the following extension Z' of Z'_T:

$$Z' := Z'_T + \frac{1}{\lambda} D_{3k} \otimes D_{3l+1}.$$

Then Z' restricts to Z'_T on $(T_{J_0})_{\mathbb{Q}} \to (T_{J_1})_{\mathbb{Q}}$ and maps D_{3k} to $\frac{1}{\lambda}\langle D_{3k}, D_{3k}\rangle D_{3l+1}$ for some λ. If D_{3k} maps isometrically to a rational multiple of D_{3l+1} i.e. if $\lambda^2 = \langle D_{3k}, D_{3k}\rangle.\langle D_{3l+1,3l+1}\rangle$, then by Witt's theorem (see [L]), Z' extends to an isometry $Z : \mathrm{H}^2(X_{3k}, \mathbb{Q}) \to \mathrm{H}^2(X_{3l+1}, \mathbb{Q})$. In this case $Z : (D_{3k})_{\mathbb{Q}} \to (D_{3l+1})_{\mathbb{Q}}$, hence $(D_{3k}^{\perp})_{\mathbb{Q}}$ maps to $(D_{3l+1}^{\perp})_{\mathbb{Q}}$, that is we get an isometry of $\mathbb{Q}$ vector spaces: $Z_T : (T_{X_{3k}})_{\mathbb{Q}} \to (T_{X_{3l+1}})_{\mathbb{Q}}$. As before consider the period point $\omega_{X_{3k}} \in (T_{X_{3k}})_{\mathbb{C}}$ of X_{3k}. Since Z_T is an isometry its image $Z_T(\omega_{X_{3k}})$ will lie in the period space. By surjectivity of the period mapping it is the period point of some other K3 surface in $\mathcal{X}_{3l+1}$. We choose that K3 surface X_{3l+1} to be the corresponding partner of X_{3k}. Then for this choice of X_{3l+1}, $Z_T : (T_{X_{3k}})_{\mathbb{Q}} \to (T_{X_{3l+1}})_{\mathbb{Q}}$. In this way we get a family of corresponding pairs (X_{3k}, X_{3l+1}).

However in order for this to work we need to solve for λ such that $\lambda^2 = \langle D_{3k}, D_{3k}\rangle.\langle D_{3l+1}, D_{3l+1}\rangle$. This imposes some constraints on k and l. In other words we need to solve $\lambda^2 = (6k-2)(6l)$ which reduces to solving $n^2 = (3k-1)(3l)$, since we can then just take $\lambda = 2n$. Given any k, the integer $3k-1$ has a unique prime factorisation. Notice that it will not contain 3 as a factor since $3k-1 \cong 2(\mod 3)$. We separate the even powers and let ρ denote the product of the remaining factors. Then choosing $l = 3\rho d^2$ (where $d \in \mathbb{Z}$) makes $(3k-1)(3l)$ a perfect square. Since for any k there are infinitely many such l we get infinitely many families of correspondences.

In a similar way one shows the existence of loci of correspondences in $\mathcal{X}_{3m+2} \times \mathcal{X}_{3l+1}$. It amounts to solving for m and n such that $(3m+2)(3l) = n^2$. The previous argument can be applied to show that such m and n exist.

□

Note that we don't get correspondences between generic elements of $\mathcal{X}_{3k}, \mathcal{X}_{3m+2}$. For those to exist we need $(3k-1)(3m+2) = n^2$. The RHS can only be congruent to 0 or 1 ($\mod 3$) whereas the left hand side is equal to 2 ($\mod 3$), so there are no solutions.

The genus one fibrations of the above example all had index 3. We can get correspoondences based on fibrations of index 2 as well. Let $\mathcal{J}_3$ be the 18 dimensional family of K3s realised as double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along a $(4,4)$ curve. Then a generic element $J_3 \in \mathcal{J}_3$ has index 2 and embeds

in $\mathbb{P}^{4m+5}$ for $m \geq 1$. Let $\mathcal{Y}_{4m+5}$ denote the family of algebraic K3s in $\mathbb{P}^{4m+5}$. By the same arguments as in the proof of Theorem 4.2 one can show the following.

Theorem 4.3. *There are infintely many pairs of integers (k, m) for which there are 19 dimensional families of correspondences in*

$$\mathcal{X}_{3k} \times \mathcal{Y}_{4m+5}, \quad \mathcal{X}_{3k+1} \times \mathcal{Y}_{4m+5}, \quad \textit{and} \quad \mathcal{X}_{3k+2} \times \mathcal{Y}_{4m+5}.$$

Note. *The (k, m) for each family will be different.*

The question of whether these $\mathbb{Q}$ isometries correspond to appropriate elements α and β in the Brauer groups of the corresponding surfaces is currently under investigation.

5 A classical correspondence

In this chapter we consider the moduli space of sheaves on a K3 surface. We present the results of Mukai and Căldăraru, and prove a theorem pertaining to a classical correspondence.

5.1 Moduli spaces

Definition 5.1. *Let Y be a surface, $\mathcal{O}_Y(1)$ be an ample class on Y, M the moduli space of $\mathcal{O}_Y(1)$-semistable sheaves with prescribed Chern classes. Then M is a **fine moduli space** if there exists a universal sheaf $\mathcal{E}$ on $Y \times M$ such that $\mathcal{E}_{|Y\times m} \cong E_m$, where E_m represents an element of the isomorphism class of sheaves corresponding to the point $m \in M$.*

In general M is not a fine moduli space, i.e. a universal sheaf does not exist on $Y \times M$. However, another sheaf closely related to the universal sheaf does exist.

Definition 5.2. *Let Y and M be as in the definition above. Then a quasi-universal sheaf is a sheaf $\mathcal{E}$ on $Y \times M$, flat over M such that if $m \in M$ represents the isomorphism class of the sheaf E_m on Y, then $\mathcal{E}_{|Y\times m} \cong E_m^n$ for some $n > 0$.*

Mukai proved that a quasi-universal sheaf associated to a moduli problem always exists. (See [M1], Appendix 6).

In the next section, we will explicitly construct the quasi-universal sheaf for a classical moduli problem.

Consider $\mathcal{E}$ on $Y \times M$ and let p_Y, p_M denote the first and the second projections. Set

$$Z_{\mathcal{E}} := \frac{p_Y^*(\sqrt{\mathrm{td}_Y}) \cdot \mathrm{ch}(\mathcal{E}) \cdot p_M^*(\sqrt{\mathrm{td}_M})}{n},$$

where n is as in Definition 5.2.

$Z_{\mathcal{E}}$ is an algebraic cycle on $Y \times M$ and induces a homomorphism

$$\begin{array}{rcccc} Z_{\mathcal{E}} & : & \mathrm{H}^*(X, \mathbb{Q}) & \longrightarrow & \mathrm{H}^*(M, \mathbb{Q}) \\ & & t & \mapsto & p_{M*}(p_Y^*(t) \cdot Z_{\mathcal{E}}) \end{array}$$

Definition 5.3. *The **Mukai vector** v associated to a sheaf F on a K3 surface Y is given by $v(F) := \mathrm{ch}(F) \cdot \sqrt{\mathrm{td}_Y}$. It decomposes as a sum $v = a^0 + a^2 + a^4$, where $a^i \in \mathrm{H}^{2i}$, also denoted by (a^0, a^2, a^4) at times.*

Definition 5.4. *The **extended Mukai lattice** is defined to be $\widetilde{\mathrm{H}}(Y, \mathbb{Z}) := \mathrm{H}^0(Y, \mathbb{Z}) \oplus \mathrm{H}^2(Y, \mathbb{Z}) \oplus \mathrm{H}^4(Y, \mathbb{Z})$ with the pairing*

$$\langle (a^0, a^2, a^4), (b^0, b^2, b^4) \rangle = \langle a^2, b^2 \rangle - \langle a^0, b^4 \rangle - \langle a^4, b^0 \rangle$$

where $\langle , \rangle$ stands for the cup product. The extended Mukai lattice has a natural weight 2 Hodge structure

$$\begin{array}{rcl} \widetilde{\mathrm{H}}^{2,0}(Y, \mathbb{C}) & := & \mathrm{H}^{2,0}(Y, \mathbb{C}) \\ \widetilde{\mathrm{H}}^{1,1}(Y, \mathbb{C}) & := & \mathrm{H}^0(Y, \mathbb{C}) \oplus \mathrm{H}^{1,1}(Y, \mathbb{C}) \oplus \mathrm{H}^4(Y, \mathbb{C}) \\ \widetilde{\mathrm{H}}^{0,2}(Y, \mathbb{C}) & := & \mathrm{H}^{0,2}(Y, \mathbb{C}) \end{array}$$

Definition 5.5. *An element v in $\widetilde{\mathrm{H}}(Y, \mathbb{Z})$ is a **primitive Mukai vector** if v is not a multiple of any other element of $\widetilde{\mathrm{H}}(Y, \mathbb{Z})$. The vector v is called isotropic if $\langle v, v \rangle$=0.*

Remark 5.1. *Another characterization of n is as follows: Let v be a primitive Mukai vector in $\widetilde{\mathrm{H}}(Y, \mathbb{Z})$. Then $n = \gcd\{\langle \omega, v \rangle\}$, where ω runs over all elements of $\widetilde{\mathrm{H}}^{1,1}(Y, \mathbb{Z})$. For a proof, see [M1], Appendix 6.*

Definition 5.6. *Let Y be an n-dimensional algebraic variety. Let $\mathcal{O}_Y(1)$ be an ample class. For a coherent sheaf F, let $r(F)$ denote the rank of F. A torsion-free coherent sheaf E on Y is **(semi)stable** with respect to $\mathcal{O}_Y(1)$ in the sense of Gieseker if for every proper non-zero subsheaf F of E, the following inequality holds*

$$h^0(E \otimes \mathcal{O}_Y(k))/r(E) > h^0(F \otimes \mathcal{O}_Y(k))/r(F)$$

for $k \gg 0$.

Now we can formulate a result due to Mukai (see [M2]).

Theorem 5.1. *Let Y be an algebraic K3 surface and v a primitive isotropic Mukai vector. Let M be the moduli space of $\mathcal{O}_Y(1)$-semistable sheaves with Mukai vector v. Assume that M is nonempty and compact. Then M is a K3 surface and the cycle $Z_{\mathcal{E}}$ induces a Hodge isometry $Z_{\mathcal{E}} : v^{\perp}/\mathbb{Z}\,v \to \mathrm{H}^2(M, \mathbb{Z})$.*

The transcendental lattice T_Y is contained in $v^{\perp}$ and $T_Y \cap \mathbb{Z}\,v = 0$, therefore $v^{\perp}/\mathbb{Z}\,v$ contains a lattice isomorphic to T_Y. From the arguments in the proof of Proposition 6.4 in [M2], it follows that $Z_{\mathcal{E}}$ embeds T_Y in T_M as the kernel of some nontrivial element $\alpha \in T_M^{\vee} \otimes \mathbb{Q}/\mathbb{Z}$. In other words, we have the following exact sequence

$$0 \to T_Y \xrightarrow{Z_{\mathcal{E}}} T_M \xrightarrow{\alpha} \mathbb{Z}/n\mathbb{Z} \to 0,$$

where n is as in Definition 5.2.

Let Y be a generic element of the family $\mathcal{Y}$ of degree 8 surfaces in $\mathbb{P}^5$. Then Y is a transverse intersection of three smooth quadric hypersurfaces Q_0, Q_1, Q_2. We consider the moduli space of $\mathcal{O}_Y(1)$-semistable sheaves on Y with the following invariants $c_0 = 2, c_1 = \mathcal{O}_Y(1), c_2 = 4$. The Mukai vector of a sheaf on Y with the above invariants is $v = (2, \mathcal{O}(1), 2)$. Since $\mathcal{O}_Y(1)$ is primitive, v is also primitive and $\langle v, v\rangle = \mathcal{O}(1)\cdot\mathcal{O}(1) - 2\cdot 2 - 2\cdot 2 = 8 - 8 = 0$. It follows by Mukai's theorem that if M is non-empty and compact, then M is a K3 surface.

We will see later that M is isomorphic to a double cover of $\mathbb{P}^2$ branched along a sextic.

We prove the following theorem:

Theorem 5.2. *Let Y be a generic intersection of three smooth quadric hypersurfaces in $\mathbb{P}^5$, M the moduli space of $\mathcal{O}_Y(1)$-semistable sheaves on Y with Chern classes $c_0 = 2, c_1 = \mathcal{O}_Y(1), c_2 = 4$, then there exists a rank 4 bundle $\mathcal{E}$ on $Y \times M$ such that $\mathcal{E}_{|Y\times m} \cong E_m \oplus E_m$, where E_m represents an element of the isomorphism class of sheaves corresponding to $m \in M$.*

In certain special cases when the rank of $\mathrm{Pic}(Y) \geq 2$, for example when Y contains a line, M is a fine moduli space, i.e. a universal sheaf exists on $Y \times M$. Before we formulate the actual construction, we need some preliminaries.

5.2 Preliminaries

Let Y be the triple intersection of three smooth quadric hypersurfaces Q_0,Q_1 and Q_2 in $\mathbb{P}^5$ with the assumption that every quadric containing Y is of rank ≥ 5. This is generically the case.

Let $N := \{Q : Q := a_0Q_0 + a_1Q_1 + a_2Q_2 = 0 \,\text{in}\, \mathbb{P}^5\}$ be the net of quadrics in $\mathbb{P}^5$ spanned by Q_0, Q_1, Q_2. Then the set of singular members of N corresponds to $N_0 := \{Q : Q := a_0Q_0 + a_1Q_1 + a_2Q_2 = 0 | \det(a_0A_0 + a_1A_1 + a_2A_2) = 0\}$ where A_i is the 6×6 symmetric matrix corresponding to the quadric Q_i. Since $\det(a_0A_0 + a_1A_1 + a_2A_2) = 0$ is a homogeneous polynomial of degree 6 in the variables a_0, a_1, a_2, the locus N_0 is a sextic curve in $N \cong \mathbb{P}^2$.

Since we assumed that every quadric containing Y has rank ≥ 5, N_0 is a smooth sextic curve. For a proof, see [T], Chapter 2, Section 3.

It turns out that the moduli space M of $\mathcal{O}_Y(1)$-semistable sheaves on Y with Mukai vector $(2, \mathcal{O}_Y(1), 2)$ is isomorphic to the double cover of $N \cong \mathbb{P}^2$, branched along N_0 (see [M2]).

We will explain below the relationship between vector bundles on Y with Mukai vector $(2, \mathcal{O}_Y(1), 2)$ and points on M. Readers familiar with the material can skip to 5.3.

Let G be the Plucker embedding of $G(1, 3)$, the Grassmannian of two planes in $\mathbb{C}^4$ or equivalently lines in $\mathbb{P}^3$. It is isomorphic to the zero locus of the quadratic equation $2z_0z_5 - z_1z_4 + z_2z_3 = 0$, where z_i are the homogeneous co-ordinates on $\mathbb{P}^5$. In G there exist two families of 2-planes corresponding to the Schubert cycles

$$\begin{array}{rcll} \sigma(p) & = & \{\ell \subseteq \mathbb{P}^3 : p \in \ell\} & \text{where } p \text{ is a point in } \mathbb{P}^3 \\ \sigma(h) & = & \{\ell \subseteq \mathbb{P}^3 : \ell \subseteq h\} & \text{where } h \text{ is a hyperplane in } \mathbb{P}^3. \end{array}$$

Let y be the point in G corresponding to the line ℓ_y in $\mathbb{P}^3$. Then $T_y(G) \cap G = \{\text{ lines in } G \text{ passing through } y\}$ which is equal to $\bigcup_{p \in \ell_y} \sigma(p) = \bigcup_{\ell \subset h} \sigma(h)$ (see [GH]).

In fact $T_y(G) \cap G$ is a cone over a smooth quadric surface $\overline{Q}$ in $\mathbb{P}^3$. Since such a surface has two distinct rulings of lines on it, it follows that there are two families of 2-planes in G passing through y, each obtained by joining the lines of one ruling with y. We can find an interpretation of this in terms of Schubert cycles as well. The cohomology group $\mathrm{H}^4(G, \mathbb{Z})$ is generated by the Schubert cycles $\sigma(p)$ and $\sigma(h)$. Then using various formulae for calculating intersections of Schubert cycles, we get that for $p_1 \neq p_2$ and $h_1 \neq h_2$, $\langle \sigma(p_1), \sigma(p_2) \rangle = 1 = \langle \sigma(h_1), \sigma(h_2) \rangle$ and for for $i = 1, 2$, $j = 1, 2$, $\langle \sigma(p_i), \sigma(h_j) \rangle = 0$ (see [GH]). Since 2-planes of opposite rulings intersect in a line, while those of the same family intersect at the point y, it follows that the 2-planes spanned by y and the lines of one ruling must be all the Schubert cycles $\sigma(p)$ for $p \in l_y$ while those spanned by the lines of the other ruling must be all the cycles $\sigma(h)$ such that $\ell \subset h$.

Lemma 5.1. *Let y be a point in a smooth quadric hypersurface Q in $\mathbb{P}^5$. Consider one of the two families of 2-planes contained in the quadric. Then the set of 2-planes in that family which pass through y form a smooth conic curve in $G(2,5)$.*

Proof. We have already shown that on G such a family arises from the span of y and the lines of a ruling on a smooth quadric surface $\overline{Q}$.

Any two quadrics in $\mathbb{P}^5$ having the same rank are isomorphic via the action of $\mathbb{P}GL(6)$. So there exists an element of $\mathbb{P}GL(6)$ mapping Q to G. Since $\mathbb{P}GL(6)$ preserves linear subspaces, the element under consideration maps the families of 2-planes contained in Q to families of 2-planes in G. So it suffices to consider only G.

Consider now $C = \{\ell \subseteq \mathbb{P}^3 : \ell \subseteq \overline{Q}\}$. This is a one dimensional variety with two smooth components C_1, C_2. Our family of 2-planes is isomorphic to one of the two components. To find the degree of C_i we take its intersection with a hyperplane class $\sigma(\ell_0)$ of $G(1,3)$. So $\sigma(\ell_0) \cap \overline{Q} = \{\ell \subset \overline{Q} : \ell \cap \ell_0 \neq \emptyset\}$. A general line ℓ_0 will intersect $\overline{Q}$ in two distinct points. One line from each ruling passes through each point, therefore the degree of each C_i is 2, which proves the lemma. □

Note. *In the case that Q is singular at a point, and x is still a smooth point, we have that $T_y(Q) \cap Q$ is a cone over a quadric cone A in $\mathbb{P}^3$. In that case there is only one ruling of lines on A and hence only one family of 2-planes contained in Q and passing through y. So C as above has only one component. Again we take the intersection of $\sigma(\ell_0)$ with A, which is equal to $\{\ell \subset A : \ell \cap \ell_0 \neq \emptyset\}$. A general line ℓ_0 will intersect A in 2 points and therefore the degree of C in this case is also 2.*

Consider the universal sequence on G

$$0 \to S \to \mathbb{C}^4 \to Q \to 0.$$

Let S_y, Q_y denote the fibres over x of S and Q respectively.

Note. *Here Q stands for the universal quotient bundle on the Grassmannian G and not a quadric hypersurface in $\mathbb{P}^5$. We are using the same notation as in [GH].*

Consider the incidence correspondence I_1 in $Y \times G(2,5)$, where

$$I_1 = \{(y, \Lambda) : y \in \Lambda, \Lambda \in \sigma(h)\ , \ell_y \subset h\}.$$

Then $I_1 \to Y$ is a smooth $\mathbb{P}^1$ bundle. Over each x the fibre is $\{h \subseteq \mathbb{P}^3 : \ell_y \subset h\}$, which is isomorphic to $\mathbb{P}(\mathbb{C}^4/S_y)$, and so $I_1 = \mathbb{P}(Q)$. Similarly we define

$$I_2 = \{(y, \Lambda) : \Lambda = \sigma(p), p \in \ell_y\}.$$

I_2 is a $\mathbb{P}^1$ bundle where now the fibre over each y is the set of points on l_y, i.e. the set of hyperplanes in ℓ_y. But this is just $\ell_y^\vee = \mathbb{P}(S_y^\vee)$. So in fact $I_2 = \mathbb{P}(S^\vee)$. $S^\vee$ and Q are rank 2 vector bundles on G which when restricted to Y have Mukai vector $(2, \mathcal{O}(1), 2)$.

Since all the smooth quadrics in $\mathbb{P}^5$ are isomorphic to G we obtain two families of 2-planes for each quadric. In the case of a singular quadric (of rank at least 5), the two families degenerate into one. Consider the net of quadrics generated by three smooth quadrics Q_0, Q_1, Q_2. The family of 2-planes sweeping out the quadrics in the net forms a double cover of $N \cong \mathbb{P}^2$ branched along a smooth sextic curve N_0. The curve N_0 represents the locus of singular quadrics in the net. Associated to each family of 2-planes in a quadric is a $\mathbb{P}^1$-bundle which is the projectivization of a rank 2 vector bundle on Y with Mukai vector $(2, \mathcal{O}(1), 2)$. It can be shown that these represent all the isomorphism classes of such bundles, (see [N]). This shows that M is isomorphic to the double cover of $\mathbb{P}^2$ branched along the sextic. In the case when the rank of every quadric containing Y is not ≥ 5, the locus of singular quadrics with rank ≤ 4 is a codimension one subset of N_0. In that case, we get a rational map from the double cover of $\mathbb{P}^2$ (branched along the sextic curve N_0) to M. Mukai's result (see [M1], [M2]) then shows that in fact M is isomorphic to the double cover of $N \cong \mathbb{P}^2$ branched along N_0.

Claim. *The generic M is not a fine moduli space.*

Proof. This follows from a result by Newstead,(see[N]). Generally $\mathrm{Pic}(M) \cong \mathbb{Z}\, D$, generated by the pull back of $\mathcal{O}(1)$ from $\mathbb{P}^2$. (Recall that M is a double cover of $\mathbb{P}^2$ branched over a sextic). Then by adjunction $\langle D, D\rangle = 2$. Now suppose M is a fine moduli space i.e., there exists a rank 2 sheaf F on $Y \times M$ satisfying the desired properties. For each $y \in Y, F_y := F_{|_{Y \times M}}$ is a rank 2 bundle with $c_1(F) = kD$ for some k. Newstead shows that for ${F_y}_{|_D}$ such that $\mathbb{P}(F_y)_{|_D} = I_{|_D}$, ${F_y}_{|_D}$ has odd degree. But that's a contradiction because degree of ${F_y}_{|_D} = \langle c_1(F_y), D\rangle = k\langle D, D\rangle = 2k$ which is an even number. □

5.3 The construction

We would like to construct a rank 4 vector bundle V on $Y \times M$ such that: $V_{|_{Y\times m}} \cong (V_1 \oplus V_2)_{|_{Y\times m}}$, and $\mathbb{P}({V_i}_{|_{Y\times m}})$ is isomorphic to the $\mathbb{P}^1$ bundle on X

associated to the family $m \in M$ of 2-planes sweeping out a quadric in the net. (See Definition 5.2.)

Definition 5.7. *We consider the incidence variety I in $Y \times M \times G(2,5)$ where $I = \{(y, m, \Lambda) : y \in \Lambda, \Lambda \in m\}$. Then $\pi : I \to Y \times M$ is a $\mathbb{P}^1$ bundle, where the fibre over each (y, m) is a smooth conic curve denoted by C. The pull back of $\mathcal{O}(1)$ from $G(2,5)$ to I is a line bundle L such that $L_{|_C} \cong \mathcal{O}_{\mathbb{P}^1}(2)$.*

Lemma 5.2. *We have the following short exact sequence*

$$0 \to K \to \pi^* \pi_* L \to L \to 0,$$

where the last map is the evaluation map.

Proof. It is enough to show this fibrewise, i.e. we need to show that the evaluation map $\pi^* \pi_* L_{|_C} \cong \mathcal{O}_C \otimes \mathrm{H}^0(\mathcal{O}(2)) \to L_{|_C}$ is surjective. If not then there exists a point p in C such that $\sigma_0(p) = \sigma_1(p) = \sigma_2(p) = 0$ where $\sigma_0, \sigma_1, \sigma_2$ are a basis for $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(2))$. But that implies $h^0(\mathbb{P}^1, \mathcal{O}(2)) = h^0(\mathbb{P}^1, \mathcal{O}(1))$. By Riemann Roch we get that $h^0(\mathbb{P}^1, \mathcal{O}(2)) = 3$ and $h^0(\mathbb{P}^1, \mathcal{O}(1)) = 2$, so we have a contradiction. $\square$

Now K is a rank 2 bundle on I such that $K_{|_C}$ has degree -2. Since all vector bundles on $\mathbb{P}^1$ are decomposable we get that $K_{|_C} \cong \mathcal{O}_C(a) \oplus \mathcal{O}_C(b)$ where $a + b = -2$

The long exact cohomology sequence associated to the sequence above in the lemma is as follows: $0 \to \mathrm{H}^0(K_{|_C}) \to \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(2)) \to \mathrm{H}^0(L_{|_C}) \to \mathrm{H}^1(K_{|_C}) \to \cdots$

Since the second map is an isomorphism we get that $\mathrm{H}^0(K_{|_C}) = 0$, which implies that $K_{|_C} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Hence $K^\vee_{|_C} \cong \mathcal{O}_{|_{\mathbb{P}^1}}(1) \oplus \mathcal{O}_{|_{\mathbb{P}^1}}(1)$.

Let $V = \pi_*(K^\vee)$. Then V is a rank 4 vector bundle on $Y \times M$ such that $V_{|Y \times m} \cong (V_1 \oplus V_2)_{|Y \times m}$ where $V_{1|Y \times m} \cong V_{2|Y \times m}$ and the restriction of $\mathbb{P}(V_i)$ to (y, m) is isomorphic to C which is the fibre of I over (y, m). So V is a quasi-universal sheaf (up to isomorphism), associated to the moduli space of sheaves on X with Mukai vector $(2, \mathcal{O}(1), 2)$.

5.4 Application

This example lends itself to a nice application of a result by Căldăraru.

Recall that we had an embedding

$$0 \to T_Y \to T_M \to \mathbb{Z}/n\mathbb{Z} \to 0$$

where M is the moduli space of sheaves on Y as before. So T_Y is the kernel of some element $\alpha \in T_M^\vee \otimes \mathbb{Q}/\mathbb{Z} = \mathrm{Br}(M)$.

Notation. *For $\omega \in \widetilde{\mathrm{H}}(M,\mathbb{Q})$, the functional $\langle \omega, \cdot \rangle_{|T_M}$ mod $\mathbb{Z}$ as an element of $\mathrm{Hom}(T_M, \mathbb{Q}/\mathbb{Z}) = \mathrm{Br}(M)$ will be denoted by $[\omega]$.*

Căldăraru has proved the following generalization of Mukai's result:

Theorem 5.3. *Let Y be an algebraic K3 surface. Let v be a primitive isotropic Mukai vector, and let M be the moduli space of $\mathcal{O}_Y(1)$-semistable sheaves with Mukai vector v. Let $n = \gcd\{\langle \omega, v \rangle\}$ where ω runs over all elements of $\widetilde{\mathrm{H}}^{1,1}(Y,\mathbb{Z})$. Assume that M is compact and nonempty (so M is a K3 surface) and by Mukai's results we have the map $Z_{\mathcal{E}} : \widetilde{\mathrm{H}}(Y,\mathbb{Q}) \to \widetilde{\mathrm{H}}(M,\mathbb{Q})$. If $u \in \widetilde{\mathrm{H}}(Y,\mathbb{Z})$ such that $\langle u, v \rangle = 1 \pmod n$ then*

1. *$\alpha := [Z_{\mathcal{E}}(u)] \in \mathrm{Br}(M)$ has n torsion and is the obstruction to the existence of a universal sheaf on $Y \times M$,*
2. *The derived category of sheaves on Y, $\mathcal{D}(Y)$, is equivalent to the derived category of α-twisted sheaves on M, $\mathcal{D}(M,\alpha)$, i.e.*

$$\mathcal{D}(Y) \cong \mathcal{D}(M,\alpha).$$

Note. *If $n = 1$, we get that the moduli space is fine. In our case, since $v = (2, \mathcal{O}(1), 2)$ and since for any $\omega = (r, \mathcal{O}(k), s)$ in $\widetilde{\mathrm{H}}^{1,1}(Y,\mathbb{Z})$, $\langle v, \omega \rangle = 8k - 2(r+s)$, we get that $n = 2$. This is another explanation for why the moduli space in general is not fine.*

We can find an explicit representative of the obstruction class u, by moving away from the generic element in the family $\mathcal{Y}$, to one where rank $\mathrm{Pic}(Y) \geq 2$.

Theorem 5.4. *Let Y be a degree 8 surface in $\mathbb{P}^5$, such that either Y contains a line L, or Y is an element of $\mathcal{M}_\beta$. Then the moduli space associated to the Mukai vector $(2, \mathcal{O}_Y(1), 2)$ is actually a fine moduli space.*

Proof. We first consider the case when Y contains a line L. The set of all such surfaces forms a codimension one locus in the family of all degree 8 surfaces in $\mathbb{P}^5$. Let $u := (0, L, 0)$, then $\langle u, v \rangle = 1$ so the moduli space is fine.

A more geometric way to see this is the following:

$I \to Y \times M$ lifts to a rank 2 vector bundle F if and only if there exists a section $\sigma : Y \times M \to I$.

Recall that in terms of Schubert cycles, the hyperplane class, $H = \mathcal{O}_Y(1)$ corresponds to $\sigma(\ell_0) = \{\ell_y : \ell_y \cap \ell_0 \neq \emptyset\}$ where ℓ_0 is a general line in $\mathbb{P}^3$ and ℓ_y is the line corresponding to $y \in Y$ (see earlier section).

The line $L \subset Y$ corresponds to the Schubert cycle $\sigma(p_0, h_0) = \{\ell : p_0 \in \ell \text{ and } \ell \subset h_0\}$ for some point p_0 and some 2-plane h_0 containing p_0 in $\mathbb{P}^3$.

$L \cdot H = 1$ implies that for a general $y \in Y$, there exists exactly one line $\ell_{L,y}$ in $\sigma(p_0, h_0)$ such that $\ell_{L,y} \cap \ell_y \neq \emptyset$.

Consider the fibre $C = I_{y,m}$. Then either

1. $C \cong \{\sigma(p) : p \in \ell_y\}$ or
2. $C \cong \{\sigma(h) : \ell_y \subset h\}$.

We will define a section of $\pi : I \to Y \times M$, using L.

In the first case, let $\sigma(y, m)$ be the unique point in C corresponding to the 2-plane $\sigma(p')$ where $p' = \ell_y \cap \ell_{L,y}$.

In the second case, map (y, m) to the point in C, corresponding to *unique 2-plane* h_0, spanned by ℓ_y and $\ell_{L,y}$.

This defines a section $\sigma : Y \times M \to I$. It follows that $I \cong \mathbb{P}(\pi_*(\mathcal{O}(\sigma)^\vee))$ and so $\pi_*(\mathcal{O}(\sigma)^\vee)$ is a universal sheaf.

This is in agreement with Căldăraru's result. As before, let L denote the class of the line and let $u = (0, L, 0)$, then $\langle u, v\rangle = 1$, so $\underline{gcd}\{\langle u, v\rangle\} = 1$. Then for any $u \in \widetilde{\mathrm{H}}(Y, \mathbb{Z})$ such that $\langle u, v\rangle = 1$, $Z_{\mathcal{E}}(u) \in \widetilde{\mathrm{H}}(M, \mathbb{Z})$, hence $[Z_{\mathcal{E}}(u)] \in \mathrm{Br}(M)$ is trivial. By Căldăraru's Theorem, $[Z_{\mathcal{E}}(u)]$ corresponds to the obstruction, which in this case is trivial, hence M is a fine moduli space.

Next we consider the case when Y is an element of the family $\mathcal{M}_\beta$, i.e. Y is a genus one fibration such that $\langle F, H\rangle = 3$ where F is the fibre class and H is the hyperplane class.

Let $u := (0, F, 0)$ then

$$\langle (0, F, 0), (2, \mathcal{O}(1), 2)\rangle = 3 \equiv 1 \pmod 2,$$

so $\gcd\{\langle u, v\rangle\} = 1$ where u runs over elements of $\widetilde{\mathrm{H}}^{1,1}(Y, \mathbb{Z})$, which as before implies that M is a fine moduli space. $\square$

Consider a marked deformation of Y in $\mathcal{Y}$ so that the class F becomes transcendental. Let $\alpha = [Z_{\mathcal{E}}(F)]$. Then by Căldăraru's Theorem, (see [C])

$$\mathcal{D}(Y) \cong \mathcal{D}(M, \alpha).$$

It is hoped that the above construction of the quasi-universal sheaf will lead to a better insight of Căldăraru's conjecture for other non fibred algebraic K3 surfaces.

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